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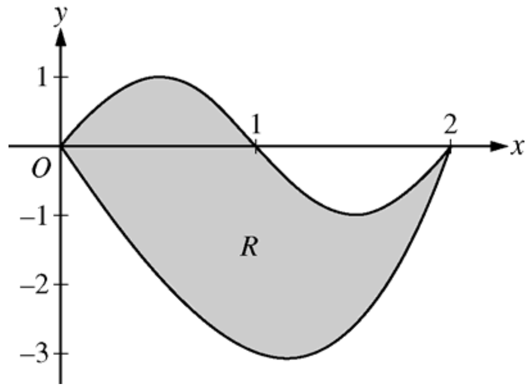
2008 SCORING GUIDELINES

Question 1



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$ Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$	$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$ The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$	$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \end{cases}$
(c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or 8.370	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

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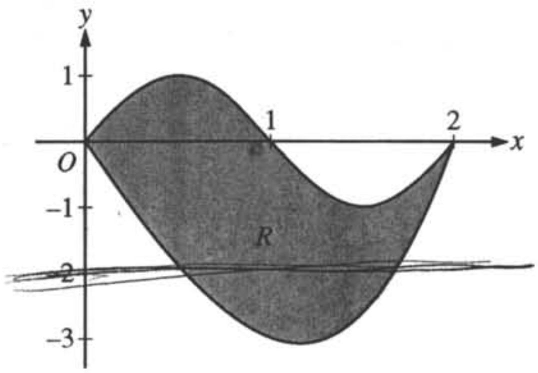
CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$\sin(\pi x) = x^3 - 4x$

$x = 2$

$A = \int_0^2 \sin(\pi x) - (x^3 - 4x)\,dx$

$A = 4$

Do not write beyond this border.

Work for problem 1(b)

$-2 = x^3 - 4x$

$x = .53918887$, and 1.6751309

$A = \int_{.539187}^{1.6751309} (-2) - (x^3 - 4x)\,dx$

$= \int_{.5391887}^{1.6751309} -2 - x^3 + 4x\,dx$

Continue problem 1 on page 5

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Work for problem 1(c)

$$V = \int_0^2 \left(\sin(\pi x) - (x^3 - 4x)\right)^2 dx$$

$V = 9.978344126$

Work for problem 1(d)

$$V = \int_0^2 \left(\sin(\pi x) - (x^3 - 4x)\right)(3 - x)\, dx$$

$V = 8.369953106$

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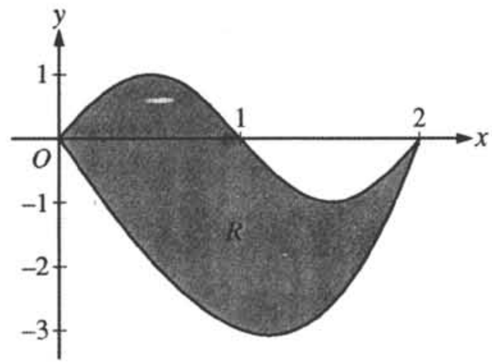
CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



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Work for problem 1(a)

$A = \sin(\pi x) - (x^3 - 4x)$

$\int_0^2 [\sin(\pi x) - (x^3 - 4x)]\, dx = 4$

$\sin(\pi x) = x^3 - 4x$

$x = -2 \quad x = 0 \quad x = 2$

Work for problem 1(b)

$\sin(\pi x) - (x^3 - 4x)$

$y = -2$

$[\sin(\pi x) - \underset{-x^3+4x}{(x^3 - 4x)}] - (-2)$

$\int_0^2 ([(\sin(\pi x) - x^3 + 4x)] + 2)\, dx$

Continue problem 1 on page 5.

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Work for problem 1(c)

cross section = square

$A = s^2$

$= [\sin \pi x - x^3 + 4x]^2$

$V = \int_0^2 [\sin \pi x - x^3 + 4x]^2 dx = 9.9783$

Work for problem 1(d)

$h(x) = 3 - x$ (depth)

$V = \pi \int_0^2 [\sin \pi x - x^3 + 4x](3-x)\,dx$

26.2950

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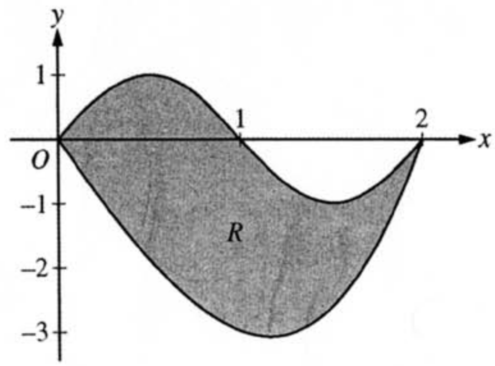
CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$$\int_0^2 (\sin(\pi x)) - (x^3 - 4x)\,dx = 4$$

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Work for problem 1(b)

$$\int_0^2 (-2) - (x^3 - 4x)\,dx$$

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Continue problem 1 on page 5.

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Work for problem 1(c)

$$V_{above} = \int_0^2 (\sin(\pi x))^2 - (x^3-4x)^2 = 8.752$$

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Work for problem 1(d)

$$V_{water\ in\ pond} = 4\pi \int_0^2 \frac{1}{2}(3-x)^2 = 17.333\pi$$

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Question 1

Overview

In this problem, students were given the graph of a region R bounded by two curves in the xy-plane. The points of intersection of the two curves were observable from the supplied graph. The formulas for the curves were given—a trigonometric function and a cubic polynomial—and students needed to match the appropriate functions to the upper and lower bounding curves. In each part, students had to set up and evaluate an appropriate integral. Part (a) asked for the area of R. Part (b) asked for the area of the portion of R below the line $y = -2,$ so students needed to use a calculator to solve for the x-coordinates of the points of intersection of $y = -2$ and the lower curve to set up the appropriate integral. Part (c) asked for the volume of a solid with base R whose cross sections perpendicular to the x-axis are squares. In part (d) students were asked to find a volume in an applied setting. They had to determine that cross sections perpendicular to the x-axis are rectangles with one dimension in region R and the other dimension supplied by $h(x) = 3 - x.$

Sample: 1A
Score: 9

The student earned all 9 points. In part (a) the student has the correct limits, integrand, and answer and earned all 3 points. In part (b) the limits are correct to three decimal places, and the limits point was earned. The student earned the integrand point on the first presentation of the integral. The second, simplified integral is also correct. In part (c) the student earned the integrand point with a correct integrand and earned the answer point since the answer is correct to three decimal places. In part (d) the integrand is correct, and the answer is correct to three decimal places.

Sample: 1B
Score: 6

The student earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student has the correct limits, integrand, and answer and earned all 3 points. In part (b) the student does not find the intersection of the cubic curve with the line $y = -2,$ and so the limits point was not earned. The integrand is not correct. In part (c) the integrand and answer are correct. The student earned both points. In part (d), although the integrand is correct, the student multiplies the integral by $\pi,$ and so the answer point was not earned.

Sample: 1C
Score: 4

The student earned 4 points: 3 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student has the correct limits, integrand, and answer and earned all 3 points. In part (b) the student earned the integrand point. The student does not find the intersection of the cubic curve with the line $y = -2,$ and so the limits point was not earned. In part (c) the student integrates a difference of squares, rather than the square of the difference of the functions. The integrand point was not earned, and the student was not eligible for the answer point. In part (d) the integrand is not correct, and the student was not eligible for the answer point.

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Question 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

(a) $L'(5.5) \approx \dfrac{L(7) - L(4)}{7 - 4} = \dfrac{150 - 126}{3} = 8$ people per hour

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The average number of people waiting in line during the first 4 hours is approximately

$$\frac{1}{4}\left(\frac{L(0) + L(1)}{2}(1 - 0) + \frac{L(1) + L(3)}{2}(3 - 1) + \frac{L(3) + L(4)}{2}(4 - 3)\right)$$

$= 155.25$ people

$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{answer} \end{cases}$

(c) L is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some t in $(1, 3)$ and some t in $(4, 7)$. Similarly, $L'(t) < 0$ for some t in $(3, 4)$ and some t in $(7, 8)$. Then, since L' is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of t in $[0, 9]$.

$3 : \begin{cases} 1 : \text{considers change in sign of } L' \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

OR

The continuity of L on $[1, 4]$ implies that L attains a maximum value there. Since $L(3) > L(1)$ and $L(3) > L(4)$, this maximum occurs on $(1, 4)$. Similarly, L attains a minimum on $(3, 7)$ and a maximum on $(4, 8)$. L is differentiable, so $L'(t) = 0$ at each relative extreme point on $(0, 9)$. Therefore $L'(t) = 0$ for at least three values of t in $[0, 9]$.

OR

$3 : \begin{cases} 1 : \text{considers relative extrema of } L \text{ on } (0, 9) \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

[*Note: There is a function L that satisfies the given conditions with $L'(t) = 0$ for exactly three values of t.*]

(d) $\int_0^3 r(t)\,dt = 972.784$

There were approximately 973 tickets sold by 3 P.M.

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$

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2A.

2 2 2 2 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Work for problem 2(a)

$L'(5.5) \approx \frac{L(7)-L(4)}{7-4} = \frac{150-126}{3} = \frac{24}{3} = 8$

8 people in line per hour

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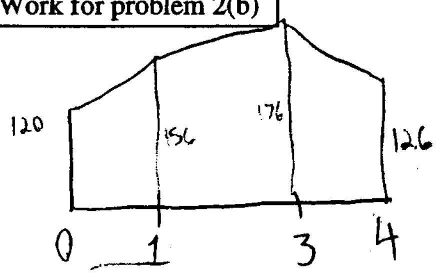
Work for problem 2(b)



$\frac{120+156}{2} + 156+176 + \frac{176+126}{2} =$

$138 + 156 + 176 + 151 = 621$

average people in line $= \frac{621}{4} = 155.25$ people

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Work for problem 2(c) Since $L(3) > L(1)$ and $L(3) > L(4)$, at some point between $t=1$ and $t=4$, the line must go from increasing to decreasing, and thus at some point $L'(t)=0$

Since $L(4) < L(3)$ and $L(4) < L7$, there must be a local minimum between $t=3$ and $t=7$, and thus another point where $L'(t)=0$

Since $L(7) > L(4)$ and $L(7) > L(8)$, there must be another local maximum between $t=4$ and $t=8$ and thus a point where $L'(t)=0$

There must be at least 3 points

Work for problem 2(d)

tickets sold $= \int_0^3 550te^{-\frac{t}{2}}\,dx = 972.784$

$\downarrow$

973

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t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

2B1

Work for problem 2(a)

$$L'(t) \approx \frac{L(7)-L(4) \text{ people}}{7-4 \text{ hour}} = \frac{150-126 \text{ people}}{3 \text{ hour}} = \frac{24 \text{ people}}{3 \text{ hour}} = \boxed{8 \ \frac{\text{People}}{\text{hour}}}$$

Work for problem 2(b)

$$\text{Average} = \frac{1}{b-a}\left(\frac{b-a}{2n}\right)\left(L(0) + 2L(1) + 2L(3) + L(4)\right)$$

$$= \frac{1}{6}\left(120 + 312 + 352 + 126\right) = \frac{910}{6} = \boxed{\frac{455}{3} \text{ People}}$$

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Continue problem 2 on page 7.

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Work for problem 2(c)

t	0	1	3	4	7	8	9
L(t)	120	156	176	126	150	80	0

<u>$L'(t)$ must equal zero at least 3 times</u>

because must equal zero whenever $L(t)$ changes from increasing to decreasing, or ~~from~~ decreasing to increasing, which we can only be sure of happening (based on the chart) between $t=3$ and $t=4$; between $t=4$ and $t=7$; and between $t=7$ and $t=8$.

Work for problem 2(d)

Tickets Sold $= \int_0^3 550\, t\, e^{-t/2}\, dt \approx \boxed{973 \text{ tickets}}$

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t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

2C1

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Work for problem 2(a)

$$\frac{f(7) - f(4)}{7-4} = \frac{150-126}{3} = 8 \ \frac{\text{people}}{\text{hr}}$$

Work for problem 2(b)

$$\frac{120+156}{2} + \frac{156+176}{4} + \frac{176+126}{2} = 372$$

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Continue problem 2 on page 7.

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$\mathcal{X}$C2

Work for problem 2(c)

2 times

Work for problem 2(d)

$r(t) = 550\, te^{-\frac{t}{2}}$ tickets/hour

$\int_0^3 550te^{-\frac{t}{2}} = 274.966$

≈ 275 tickets

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Question 2

Overview

This problem presented students with a table of data indicating the number of people $L(t)$ in line at a concert ticket office, sampled at seven times t during the 9 hours that tickets were being sold. (The question stated that $L(t)$ was twice differentiable.) Part (a) asked for an estimate for the rate of change of the number of people in line at a time that fell between the times sampled in the table. Students were to use data from the table to calculate an average rate of change to approximate this value. Part (b) asked for an estimate of the average number of people waiting in line during the first 4 hours and specified the use of a trapezoidal sum. Students needed to recognize that the computation of an average value involves a definite integral, approximate this integral with a trapezoidal sum, and then divide this total accumulation of people hours by 4 hours to obtain the average. Part (c) asked for the minimum number of solutions guaranteed for $L'(t) = 0$ during the 9 hours. Students were expected to recognize that a change in direction (increasing/decreasing) for a twice-differentiable function forces a value of 0 for its derivative. Part (d) provided the function $r(t) = 550te^{-t/2}$ tickets per hour as a model of the rate at which tickets were sold during the 9 hours and asked students to find the number of tickets sold in the first 3 hours, to the nearest whole number, using this model. Students needed to recognize that total tickets sold could be determined by a definite integral of the rate $r(t)$ at which tickets were sold.

Sample: 2A
Score: 9

The student earned all 9 points. In part (c) the student might have given a more complete justification for the existence of a local maximum on the interval $(1, 4)$. It would have been better if the student had used the word "graph" rather than "line" in the first paragraph. The response is more complete and uses better terminology in the second and third paragraphs, and the student gives the correct answer.

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), 2 points in part (c), and 2 points in part (d). The student earned both points in part (a) with a correct estimate and correct units. In part (b) the student's expression reflects subdivisions of $[0, 4]$ of equal length. The student did not earn any points. In part (c) the student earned the first point by considering $L(t)$ changing from increasing to decreasing. (The student goes on to consider $L(t)$ changing from decreasing to increasing, but this was not necessary to earn the first point.) The student did not earn the second point since it is not necessarily true that $L(t)$ changes from increasing to decreasing on the interval $[3, 4]$. The student earned the third point with the correct answer of 3. The student earned both points in part (d).

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Question 2 (continued)

Sample: 2C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), no points in part (c), and 1 point in part (d). The student earned both points in part (a) with a correct estimate and correct units. The student's use of f rather than L was not penalized. The student did not earn points in part (b) because the given expression is not a valid trapezoidal sum. In part (c) the student did not earn the first point. As a result of the incorrect answer ("2 times"), the student did not earn the other points in part (c). In part (d) the student earned the integrand point. Since the value of the integral is not correct, the student did not earn the second point, even though the limits on the integral are correct.

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Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x-2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.

$$4: \begin{cases} 2: P_1(x) \\ 1: P_1(1.9) \\ 1: P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$$

(b) $P_3(x) = 80 + 128(x-2) + \dfrac{488}{6}(x-2)^2 + \dfrac{448}{18}(x-2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

$$3: \begin{cases} 2: P_3(x) \\ 1: P_3(1.9) \end{cases}$$

(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\displaystyle\max_{1.9 \le x \le 2} \left| h^{(4)}(x) \right| = \frac{584}{9}$.

Therefore, $\displaystyle |h(1.9) - P_3(1.9)| \le \frac{584}{9} \frac{|1.9 - 2|^4}{4!}$
$= 2.7037 \times 10^{-4}$
$< 3 \times 10^{-4}$

$$2: \begin{cases} 1: \text{form of Lagrange error estimate} \\ 1: \text{reasoning} \end{cases}$$

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x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Work for problem 3(a)

$P(x) = h(2) + \frac{h'(2)}{1!}(x-2)^1$

$P(x) = 80 + 128(x-2)$

$h(1.9) \approx P(1.9) = 67.2000$

$h(1.9) \approx 67.2000$

This approximation is less than $h(1.9)$ because since h and h' are increasing, h is concave up and the linear approximation line lies below graph of h $\therefore$ the approximation is less than $h(1.9)$

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Work for problem 3(b)

$$P(x) = h(2) + \frac{h'(2)}{1!}(x-2)^1 + \frac{h''(2)}{2!}(x-2)^2 + \frac{h'''(2)}{3!}(x-2)^3$$

$$P(x) = 80 + 128(x-2)^1 + \frac{\frac{488}{3}}{2}(x-2)^2 + \frac{\frac{448}{3}}{6}(x-2)^3$$

$$h(1.9) \approx P(1.9) = 67.9884$$

$$h(1.9) \approx 67.9884$$

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Work for problem 3(c)

$$|\text{error}| < 3\times10^{-4} \qquad 3\times10^{-4} = .0003$$

$$\text{Lagrange Error} = \left|\frac{\text{Max deriv}}{(n+1)!}(x-c)^{n+1}\right| < 3\times10^{-4}$$

$$\left|\frac{h^4(x)}{4!}(1.9-2)^4\right| < 3\times10^{-4}$$

$$\left|\frac{\frac{584}{9}}{4!}(1.9-2)^4\right| < 3\times10^{-4}$$

$$.00027 < .0003$$

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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	0th der	1st d.	2nd ds	3rd ds	4th d.
x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Work for problem 3(a)

$$T(x) = \frac{f^{(0)}(x-a)^0}{0!} + \frac{f^{(1)}(x-a)^1}{1!}$$

$$T_1(x) = 80 + 128(x-2)$$

$$h(1.9) \approx 67.2$$

The first-degree Taylor approximation is an underapproximation of h(1.9) because the double-derivative of h, h''(x), at 2 is positive ∴ the function is concave up around x = 2 and the values will be increasing.

Continue problem 3 on page 9.

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Work for problem 3(b)

$$T_3^{(2)} = \frac{f^{(0)}(x-a)^0}{0!} + \frac{f^{(1)}(x-a)^1}{1!} + \frac{f^{(2)}(x-a)^2}{2!} + \frac{f^{(3)}(x-a)^3}{3!}$$

$$T_3^{(2)} = 80 + 128(x-2) + \frac{488}{3}\cdot\frac{1}{2}(x-2)^2 + \frac{448}{3}\cdot\frac{1}{6}(x-2)^3$$

$$T_3^{(2)} = 80 + 128(x-2) + \frac{244(x-2)^2}{3} + \frac{224(x-2)^3}{9}$$

$h(1.9) \approx 67.986$

Work for problem 3(c)

Langrange error bound for n^{th} degree Taylor series is the $(n+1)^{th}$ term

4^{th} term is $\frac{f^{(4)}(x-a)^4}{4!} = \frac{584(x-2)^4}{9\cdot 4!} = \frac{73}{27}(x-2)^4$

plug in $1.9 = 2.704 \times 10^{-4}$

yes, the langrange error bound for the third degree Taylor polynomial is $< 3\times10^{-4}$ and thus the error is $< 3\times10^{-4}$.

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END OF PART A OF SECTION II
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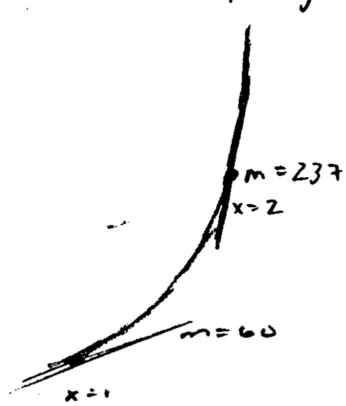
x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Work for problem 3(a)

a) $h(x) = 80 + 128(x-2)$

$h(1.9) = 80 + 128(1.9-2) = 67.200$

The approximation is less than the actual value because when comparing the slopes between 1 and 2, and, 2 and 3, they are 69 and 237 respectively. Meaning that the slopes are becoming faster, and steeper and because h is increasing, the tangent lines are under the concave up graph



Continue problem 3 on page

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$3C_2$

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Work for problem 3(b)

$$h(x) = 80 + 128(x-2) + \frac{\frac{488}{3}}{2!}(x-2)^2 + \frac{\frac{448}{3}}{3!}(x-2)^3$$

$$= 80 + 128(x-2) + \frac{244(x-2)^2}{3} + \frac{224(x-2)^3}{9}$$

$$h(1.9) = 80 + 128(1.9-2) + \frac{244(1.9-2)^2}{3} + \frac{244(1.9-2)^3}{9}$$

$$= 67.986$$

Work for problem 3(c)

approximate $h(1.9) = 67.986$

. error less than $.0003 = 3 \cdot 10^{-4}$

~~tl ~~

H has an error less than

3×10^{-4}.

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Question 3

Overview

This problem presented students with a table of values for a function h and its derivatives up to the fourth order at $x = 1$, $x = 2$, and $x = 3$. The question stated that h has derivatives of all orders, and that the first four derivatives are increasing on $1 \le x \le 3$. Part (a) asked for the first-degree Taylor polynomial about $x = 2$ and the approximation for $h(1.9)$ given by this polynomial. Students needed to use the given information to determine that the graph of h is concave up between $x = 1.9$ and $x = 2$ to conclude that this approximation is less than the value of $h(1.9)$. Part (b) asked for the third-degree Taylor polynomial about $x = 2$ and the approximation for $h(1.9)$ given by this polynomial. In part (c) students were expected to observe that the given conditions imply that $\left|h^{(4)}(x)\right|$ is bounded above by $h^{(4)}(2)$ on $1.9 \le x \le 2$ and apply this to the Lagrange error bound to show that the estimate in part (b) has error less than 3×10^{-4}.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the student gives a correct linear polynomial and a correct evaluation of $P_1(1.9)$. The student earned the first 3 points. The student states that the approximation is an underapproximation but only mentions that $h''(2) > 0$. An argument at a point was not sufficient to earn the last point. In part (b) the student's polynomial is correct and earned both points. The student incorrectly evaluates $P_3(1.9)$ so did not earn the last point. In part (c) the student has the proper form for the Lagrange error term and earned the first point. The student never bounds the fourth derivative so did not earn the last point.

Sample: 3C
Score: 4

The student earned 4 points: 2 points in part (a), 2 points in part (b), and no points in part (c). In part (a) the student gives a linear polynomial that is correctly centered but equates the polynomial to $h(x)$ and earned only 1 point. The student correctly evaluates $P_1(1.9)$ and earned 1 point. The student states that the "approximation is less than the actual value" but provides an argument that is not sufficient to earn the last point. In part (b) the student's polynomial is correct and earned both points. The student incorrectly evaluates $P_3(1.9)$ so did not earn the last point. The student was not penalized a second time for equating $h(x)$ to a polynomial. In part (c) the student does not have the proper form for the Lagrange error term so did not earn either point.

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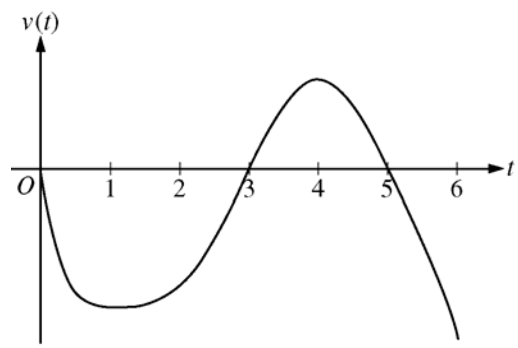
Question 4



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$. $x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$ $x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$ Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.	$3 : \begin{cases} 1 : \text{identifies } t = 3 \text{ as a candidate} \\ 1 : \text{considers } \int_0^6 v(t)\,dt \\ 1 : \text{conclusion} \end{cases}$
(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$. By the Intermediate Value Theorem, there are three values of t for which the particle is at $x(t) = -8$.	$3 : \begin{cases} 1 : \text{positions at } t = 3,\ t = 5, \\ \quad \text{and } t = 6 \\ 1 : \text{description of motion} \\ 1 : \text{conclusion} \end{cases}$
(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.	1 : answer with reason
(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

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4A

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CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

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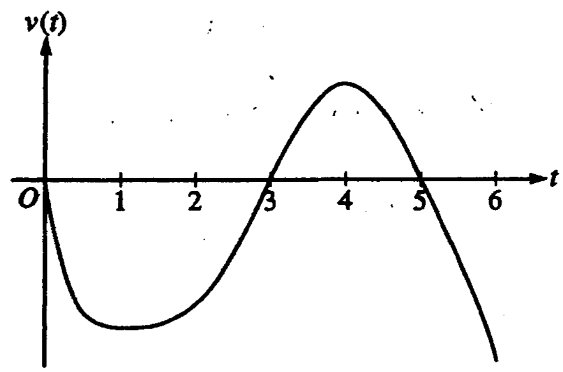

Graph of v

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Work for problem 4(a)

$x(t) = \int v(t)\,dt$

$x(0) = -2$

farthest to the left = greatest negative $x(t)$

from $[0,3]$ constantly moving to left

position: $x(0) - 8 = x(3)$

$-2 - 8 = x(3)$

$-10 = x(3)$

during $[3,5]$ moving to right

position: $x(0) - 8 + 3 = x(5)$ $\quad -2-8+3 = -7 = x(5)$

$[5,6]$ moving left

total displacement:

$x(0) - 8 + 3 - 2 =$

$-2 - 8 + 3 - 2 = -9$

farthest to the left @ $t = 3$, where $x = -10$

Work for problem 4(b)

$[0,3]$ particle moves from -2 to -10 (passes $x = -8$ once here)

$[3,5]$ moves from -10 to -7 (passes $x = -8$ once here)

$[5,6]$ from -7 to -9 passes -8

3 times particle is at 8 b/c displacement calculations crossed $x = -8$ three times.

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Work for problem 4(c)

the speed of the particle is decreasing because acceleration ($v'(t)$) and the direction of movement are in opposite directions. (acceleration is + but $v(t)$ is negative)

Work for problem 4(d)

Acceleration is negative from $[0,1) \vee (4,6]$ because the slope of velocity ($v'(t)$) is negative there.

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4B1

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CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

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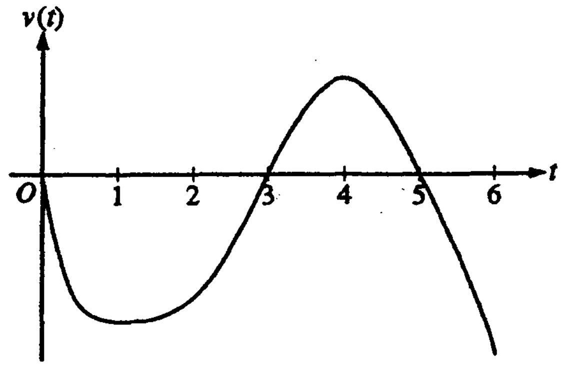

Graph of v

Work for problem 4(a) The point at which the particle is farthest to the left is $t=3$ here the particle would be at $x=-10$

This is so because the particle starts at position $x=-2$ then decreases on the graph for a total of 8 from $(0,3)$. The point farthest left could not be at any other point because after $t=3$ the velocity is positive so the particle is moving right. This positive value from $(3,5)$ has a greater area than $(5,6)$ so the particle ends farther to the right than at 3

Work for problem 4(b) On the interval $0 \le t \le 6$ $x=-8$ three times

on the interval $(0,3)$ it adds -8 to the -2 so it passes -8 on some point and continues to -10 on $(3,5)$ it adds 3 bringing it to -7 passing -8 again and finally on $(5,6)$ it adds -2 bringing it to $x=-9$ passing -8 for the third time.

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Work for problem 4(c) on the interval $2 \leq t \leq 3$ the speed of the particle is increasing this is because from (2,3) the graph has a positive slope and the graph of v is velocity so if it is a positive slope it has positive acceleration so the particle is increasing in speed.

Work for problem 4(d)

The acceleration of the particle is negative on the intervals $0 \leq t \leq .5$ and $4 \leq t \leq 6$ this is because the slope of the graph of velocity is negative at these points.

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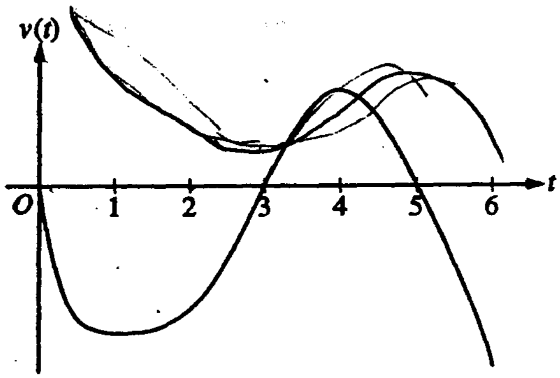
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Graph of v

Work for problem 4(a)

$\frac{dx}{dt} = v(t)$ $\frac{dy}{dt}$ dx/dt

(-2,0)

$\frac{dx}{dt} = 0$

~~at t = 1, 4~~

at $\boxed{t=3}$, it is the farthest left b/c $\frac{dx}{dt}$ goes from negative to positive

position - 8 - 2

The position is at $\boxed{x=6}$

Work for problem 4(b)

the particle is at x = -8 at t-value, t = 3 because $\int_0^x v(t)\,dt = -8$ only when x = 3

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Work for problem 4(c)

$s = |v(t)|$ the speed is increasing because the graph of $v(t)$ is increasing on the interval (2,3) and speed is the absolute value of the velocity ($a(t) = v'(t)$ is pos. on this interval.)

Work for problem 4(d)

the acceleration is negative on (0,1) and (4,6) because the velocity is decreasing on these intervals

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Question 4

Overview

This problem presented students with the graph of a velocity function for a particle in motion along the x-axis for $0 \le t \le 6$. Areas of regions between the velocity curve and the t-axis were also given. Part (a) asked for the time and position of the particle when it is farthest left, so students needed to know that velocity is the derivative of position, and they had to be able to determine positions at critical times from the particle's initial position and areas of regions bounded by the velocity curve and the t-axis. Part (b) tested knowledge of the Intermediate Value Theorem applied to information about the particle's position function derived from its initial position and the supplied graph of its derivative. Part (c) asked students to interpret information about the speed of the particle from the velocity graph: namely, that if velocity is negative but increasing, then its absolute value, speed, is decreasing. Part (d) asked for the time intervals over which acceleration is negative, so students had to recognize that acceleration is the derivative of velocity. The sign of acceleration can be read from the intervals of increase/decrease of the velocity function.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 3 points in part (a), 3 points in part (b), no points in part (c), and no points in part (d). In part (a) the student clearly uses $t = 3,$ describes the motion of the particle over each interval, and draws the correct conclusion. The student earned all 3 points. In part (b) the student finds the positions of the particle at the appropriate times, describes how the particle passes $x = -8$ on each interval, and draws the correct conclusion. The student earned all 3 points. In part (c) the student concludes that the speed of the particle is increasing so did not earn the point. In part (d) the student provides intervals that are not correct. Since the intervals do not have correct endpoints, the student did not earn any points.

Sample: 4C
Score: 3

The student earned 3 points: 1 point in part (a), no points in part (b), no points in part (c), and 2 points in part (d). In part (a) the student identifies $t = 3$ as a candidate but uses a local minimum justification. As a result, the student did not earn the last 2 points. In part (b) the student has a correct conclusion based on the work presented in part (a) but provides a reason that is not correct. The student did not earn the first point because the positions at $t = 5$ and $t = 6$ are not considered. The student does not describe the motion of the particle from position to position so did not earn the second or third points. In part (c) the student concludes that the speed of the particle is increasing so did not earn the point. In part (d) the student provides correct intervals and justification and earned both points.

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Question 5

The derivative of a function f is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$

Therefore, f has a relative minimum at $x = 3$.

$2 : \begin{cases} 1\text{: minimum at } x = 3 \\ 1\text{: justification} \end{cases}$

(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$
$f''(x) > 0$ for $x > 2$

$f'(x) < 0$ for $0 < x < 3$

Therefore, the graph of f is both decreasing and concave up on the interval $2 < x < 3$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 7 + \int_1^3 (x-3)e^x\,dx$

$$u = x - 3 \quad dv = e^x\,dx$$
$$du = dx \qquad v = e^x$$

$$f(3) = 7 + (x-3)e^x\Big|_1^3 - \int_1^3 e^x\,dx$$
$$= 7 + \left((x-3)e^x - e^x\right)\Big|_1^3$$
$$= 7 + 3e - e^3$$

$4: \begin{cases} 1 : \text{uses initial condition} \\ 2 : \text{integration by parts} \\ 1 : \text{answer} \end{cases}$

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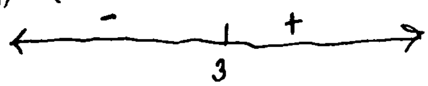
BC

5A₁

Work for problem 5(a)

$f'(x) = (x-3)e^x$



$f'(0) = -3(e^0) = -3$

$f'(5) = 2e^5$

f has a rel minimum b/c on the interval ~~of~~ of X from (0,3) the function is decreasing, or $f'(x) < 0$, and on the interval of x from $(3, \infty)$ the function is increasing or $f'(x) > 0$, which means there is a relative minimum when $x=3$.

Work for problem 5(b)

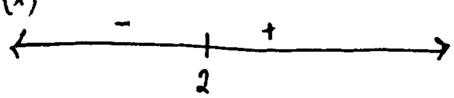
$f''(x) = e^x + (x-3)e^x$

$0 = e^x(1 + x - 3)$

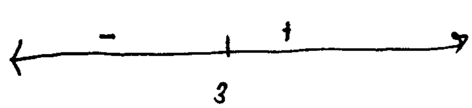
$0 = e^x(x-2)$

critical value: $x = 2$

$f''(x)$



$f'(x)$



On the x interval ~~(0,3)~~ (2,3) the graph is both decreasing and concave up b/c $f'(x) < 0$ on $x \in (0,3)$ and $f''(x) > 0$ when $x \in (2, \infty)$ and therefore the graph of f is concave up + decreasing when $x \in (2,3)$.

Continue problem 5 on page 13.

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5A2

Work for problem 5(c)

f(1)

$f(1) = 7$

$f(3) > 7$ b/c $f(x)$ is increasing from $x \in (1, \infty)$.

$f(x) = \int (x-3)e^x \, dx$ $dv = e^x$ $u = x-3$

$v = e^x$ $du = 1$

$\int u\,dv = uv - \int v\,du$

$= (x-3)e^x - \int e^x \, dx$ $e^x x - 3e^x - e^x$

$e^x x - 4e^x$

$= \boxed{(x-3)e^x - e^x + C}$ $\Rightarrow e^x(x-4) + C$

~~$2e^x - e^1 + C = 7$~~ $e^1(-3) + C = 7$

~~$e + e = 7$~~ $-3e + C = 7$

~~$C = 7e$~~ $7 + 3e = C$

~~$e^x(x-3) - e^x + 7 - e$~~ $\boxed{(x-3)e^x - e^x + 7 + 3e = f(x)}$

~~$e^3(3-3) - e^3 + 7 - e$~~ $(0)e^3 - e^3 + 7 + 3e = f(3)$

~~$0 + 7 - e - e^3 = f(3)$~~ $\boxed{7 + 3e - e^3 = f(3)}$

~~$\boxed{7 - e - e^3} = f(3)$~~

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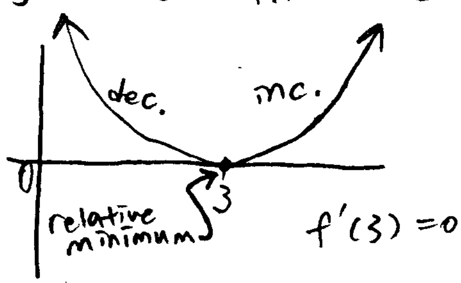
Work for problem 5(a)

The function has a relative minimum because $f'(x) < 0$ on $0 < x < 3$ and $f'(x) > 0$ on $3 < x$. This means that the function f has a relative minimum since f is decreasing on $0 < x < 3$ and increasing on $3 < x$.

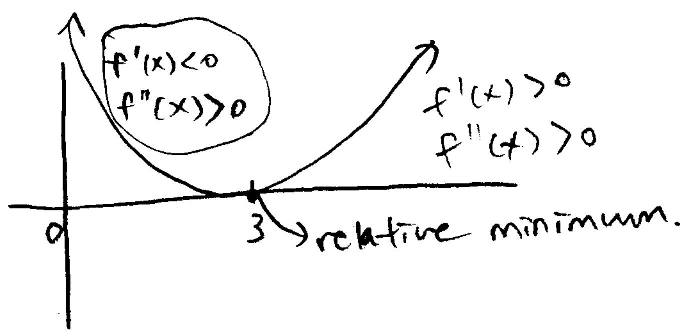
ex)



Work for problem 5(b)

The graph of f is both decreasing and concave up on $0 < x < 3$ because $f'(x) < 0$ and $f''(x) > 0$ on $0 < x < 3$.

ex)



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Work for problem 5(c)

$$f(x) = \int (x-3)e^x\,dx$$

let $u = x-3$ $\quad dv = e^x\,dx$

$du = dx$ $\quad v = e^x$

$$f(x) = e^x(x-3) - \int e^x\,dx + C$$

$$= e^x(x-3) - e^x + C$$

$$f(1) = e(-2) - e + C = 1$$

$$C = 1 + 3e$$

$$f(3) = e^3\cancel{(3-3)}^{0} - e^3 + (1+3e)$$

$$= 1 + 3e - e^3$$

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5C1

Work for problem 5(a)

$- \mid +$

3

min

AT $x=3$ There is A RELATIVE MIN BECAUSE THE FUNCTION goes FROM Decreasing TO Increasing.

Work for problem 5(b)

Dec $(0,3)$

$y'' = (x-3)(e^x) + e^x$ $\quad - \mid +$

$\cap\ 3\ \cup$

The Graph is ~~[illegible]~~ Never Decreasing AND Concave Up

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5C2

Work for problem 5(c)

$\int (x-3) e^{x} \, dx$

$u = x-3 \quad x = u+3 \quad du = dx$

$\int_1^7 (u) e^{u+3} \, du$

$\frac{1}{2} u^2 e^{u+3} \Big]_1^7$

$\frac{1}{2} 49 e^{10} - \frac{1}{2}(1) e^4 =$

$\frac{49}{2} e^{10} - \frac{1}{2} e^4$

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AP® CALCULUS BC
2008 SCORING COMMENTARY

Question 5

Overview

In this problem, students were told that a function f has derivative $f'(x) = (x - 3)e^x$ and that $f(1) = 7.$ In part (a) students needed to determine with justification the character of the critical point for f at $x = 3.$ Part (b) asked for the intervals on which the graph of f is both decreasing and concave up. For this, students had to apply the product rule to obtain $f''(x).$ In part (c) students needed to solve the initial value problem to find $f(3),$ employing integration by parts along the way.

Sample: 5A
Score: 9

The student earned all 9 points. In part (a) the student correctly identifies $x = 3$ as the relative minimum and gives a correct justification. It is important to note that had the student only written that the function changes from decreasing to increasing at $x = 3,$ the response would not have earned the justification point. The student makes the connection between the derivative being negative and then positive on either side of the critical point to earn the justification point. In part (b) the student has a correct second derivative using the product rule. The student also finds the correct interval and explains the reason for the choice. In part (c) the student uses integration by parts to find the correct antiderivative. The student uses the initial condition and gives a correct answer.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), and 4 points in part (c). In part (a) the student correctly identifies $x = 3$ as the relative minimum and gives a correct justification. In part (b) the student does not have a correct second derivative or the correct answer. In part (c) the student uses integration by parts to find the correct antiderivative. The student uses the initial condition and gives a correct answer.

Sample: 5C
Score: 3

The student earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the student correctly identifies $x = 3$ as the relative minimum but does not give a correct justification. In part (b) the student has a correct second derivative but does not give a correct answer. In part (c) the student chooses to integrate using substitution instead of integration by parts so did not earn any points.

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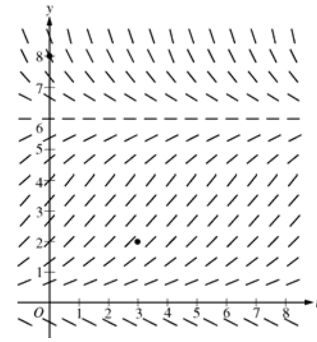
AP® CALCULUS BC
2008 SCORING GUIDELINES

Question 6

Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.
(Note: Use the axes provided in the exam booklet.)

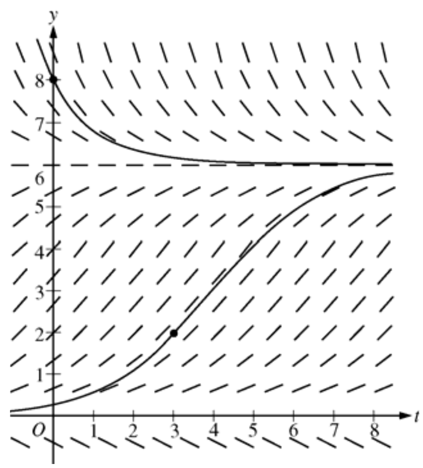


(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

(c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

(d) What is the range of f for $t \geq 0$?

(a)



$2 : \begin{cases} 1\text{: solution curve through } (0,8) \\ 1\text{: solution curve through } (3,2) \end{cases}$

(b) $f\left(\frac{1}{2}\right) \approx 8 + (-2)\left(\frac{1}{2}\right) = 7$

$f(1) \approx 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right) = \frac{105}{16}$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{approximation of } f(1) \end{cases}$

(c) $\frac{d^2y}{dt^2} = \frac{1}{8}\frac{dy}{dt}(6 - y) + \frac{y}{8}\left(-\frac{dy}{dt}\right)$

$f(0) = 8;\ f'(0) = \left.\frac{dy}{dt}\right|_{t=0} = \frac{8}{8}(6 - 8) = -2;$ and

$f''(0) = \left.\frac{d^2y}{dt^2}\right|_{t=0} = \frac{1}{8}(-2)(-2) + \frac{8}{8}(2) = \frac{5}{2}$

The second-degree Taylor polynomial for f about $t = 0$ is $P_2(t) = 8 - 2t + \frac{5}{4}t^2$.

$f(1) \approx P_2(1) = \frac{29}{4}$

$4 : \begin{cases} 2 : \frac{d^2y}{dt^2} \\ 1 : \text{second-degree Taylor polynomial} \\ 1 : \text{approximation of } f(1) \end{cases}$

(d) The range of f for $t \geq 0$ is $6 < y \leq 8$.

1 : answer

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BC 6A,

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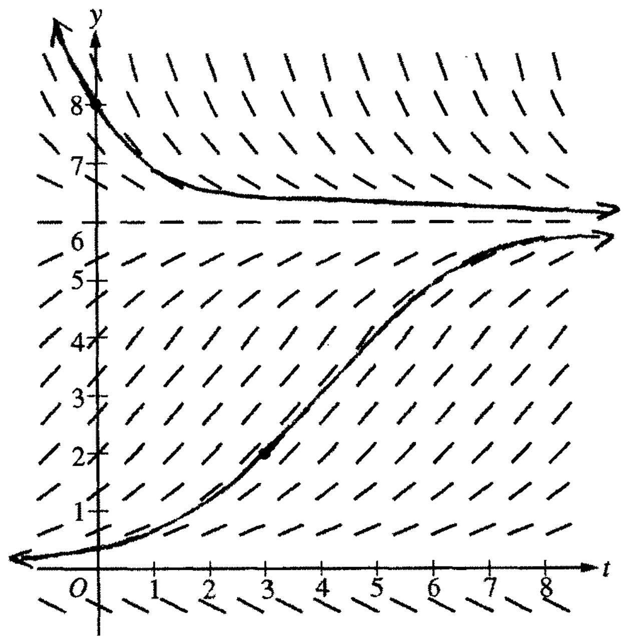
Work for problem 6(a)



Work for problem 6(b)

$\frac{dy}{dt} = \frac{y}{8}(6-y)$

$(0,8)$

$y_n = y_{n-1} + F(t_{n-1}, y_{n-1})h$

$y_1 = 8 + \left[\frac{8}{8}(6-8)\right](0.5) = 7$

$y_2 = 7 + \left[\frac{7}{8}(6-7)\right](0.5) = 7 - \frac{7}{16} = \frac{105}{16}$

$\frac{7}{16}\ \frac{16}{112}$

$\begin{array}{r} 112 \\ -\ 7 \\ \hline 105 \end{array}$

$f(1) \approx \boxed{\frac{105}{16}}$

Continue problem 6 on page 15.

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Work for problem 6(c)

$$T_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$\frac{dy}{dt} = \frac{3}{4}y - \frac{y^2}{8}$ $T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

$\left.\frac{dy}{dt}\right|_{t=0} = \frac{8}{8}(6-8)$ $T_2(x) = 8 - 2x + \frac{5}{4}x^2$

$= -2$ $T_2(1) = 8 - 2 + \frac{5}{4} = 7\frac{1}{4} = \frac{29}{4}$

$\frac{d^2y}{dt^2} = \frac{3}{4}\frac{dy}{dt} - \frac{2y}{8}\frac{dy}{dt}$ $f(1) \approx \boxed{\frac{29}{4}}$

$\left.\frac{d^2y}{dt^2}\right|_{t=0} = \frac{3}{4}(-2) - 2(-2) = -\frac{3}{2} + 4 = \frac{5}{2}$

Work for problem 6(d)

since $\lim_{t\to\infty} f(t) = 6$ and $f(0) = 8$,

the range for f on $t \geq 0$ is

$\boxed{(6, 8]}$

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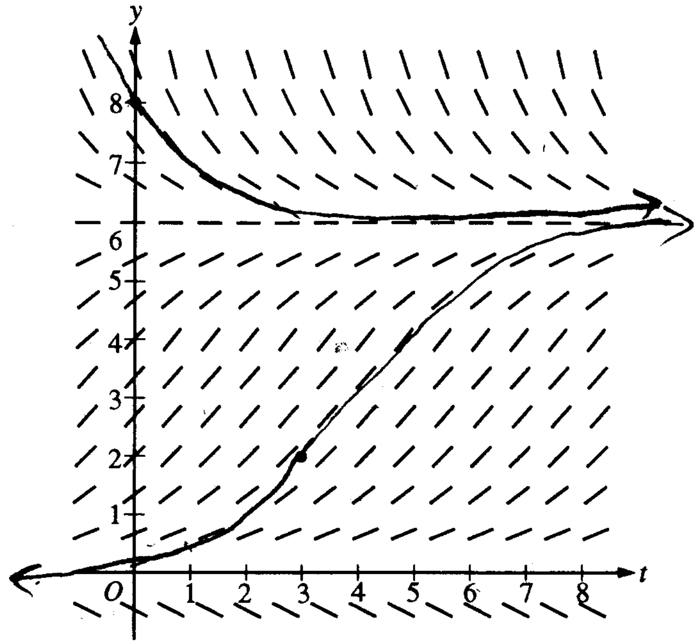
Work for problem 6(a)



Work for problem 6(b)

$\cancel{t}$	$f(\cancel{t})$
0	8
.5	7
1	$7-\frac{9}{16}$

$\frac{dy}{.5}=\frac{8}{8}(6-8)$

$dy=-2(.5)$

$dy=-1$

$\frac{dy}{.5}=\frac{9}{8}(6-7)$

$dy=\frac{-9}{8}\left(\frac{1}{2}\right)$

$dy=\frac{-9}{16}$

$\boxed{f(1)\approx\frac{103}{16}}$

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6B2

Work for problem 6(c)

$P(t) = \frac{f(0)(x)^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!}$

$P(t) = 8 + -2(x) + \frac{-3}{2}\left(\frac{x^2}{2}\right)$

$P(t) = 8 - 2x - \frac{3x^2}{4}$

$f(1) \approx P(1) = 8 - 2(1) - \frac{3(1)^2}{4}$

$f(1) \approx 8 - 2 - \frac{3}{4}$

$f(1) \approx \frac{32}{4} - \frac{8}{4} - \frac{3}{4}$

$\boxed{f(1) \approx \frac{21}{4}}$

$f(0) = 8$

$f'(0) = -2$

$f''(t) = \frac{d}{dt}\left[\frac{1}{8}y(6-y)\right]$

$f''(t) = \frac{1}{8}\frac{d}{dt}\left[6y - y^2\right]$

$f''(t) = \frac{1}{8}\left(6\frac{dy}{dt} - 2y\frac{dy}{dt}\right)$

$f''(0) = \frac{1}{8}\left[6(f'(0)) - 2(0)(f'(0))\right]$

$f''(0) = \frac{-12}{8} = -\frac{3}{2}$

Work for problem 6(d)

$\frac{dy}{dt} = \frac{y}{8}(6-y)$

$\frac{dy}{dt} = \frac{1}{8}(6y - y^2)$

$\int \frac{8}{(6-y)y}\, dy = \int dt$

$\frac{8}{(6-y)y} = \frac{A}{6-y} + \frac{B}{y}$

$8 = Ay + -By + 6B$

$A - B = 0 \qquad 6B = 8$

$A - \frac{4}{3} = 0 \qquad B = \frac{4}{3}$

$A = \frac{4}{3}$

$\frac{4}{3}\left[\int \frac{1}{6-y}\, dy + \int \frac{1}{y}\, dy\right] = t + C$

$\frac{-4}{3}\left[\ln|6-y| + \ln|y|\right] = \frac{-3}{4}(t + c)$

$e^{\left[\ln|6y - y^2|\right]} = e^{\frac{-3}{4}t + C_1}$

$-y^2 + 6y = C_1 e^{-\frac{3}{4}t}$

$(y-3)^2 = C_1 e^{-\frac{3}{4}t} + 9$

$y = \sqrt{C_1 e^{-\frac{3}{4}t} + 9} + 3$

$\lim_{t \to \infty} \sqrt{C_1 e^{-\frac{3}{4}t} + 9} + 3$

$\lim_{t \to \infty} \sqrt{9} + 3 = 6$

$range = [8, 6]$

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6C1

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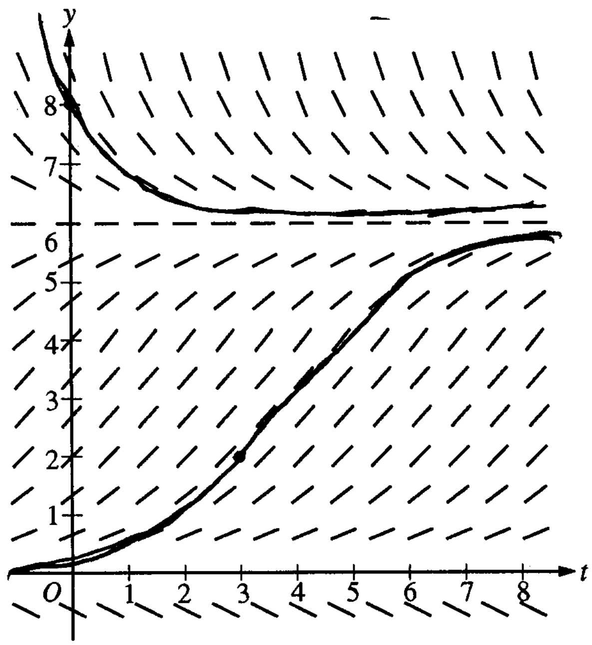
Work for problem 6(a)



Work for problem 6(b)

$(0, 8) \qquad (0.5, 0.5 \cdot \frac{dy}{dt} + 8)$

$\frac{dy}{dt}$ @ $(0, 8) = \frac{8}{8}(6-8) = -2$

$(0.5, -1+8) = (0.5, 7)$

$\frac{dy}{dt}$ @ $(0.5, 7) = \frac{7}{8}(6-7) = -\frac{7}{8}$

$(1, 7+0.5 \cdot \frac{-7}{8}) = (1, \frac{105}{16})$

$f(1) \approx \frac{105}{16} = 6.5625$

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Work for problem 6(c)

$f(c) + f'(c)(\cancel{t}-c) + \frac{f''(c)(\cancel{t}-c)^2}{2}$ @ $c=0$

$f(0) + f'(0)(t) + \frac{f''(0)\cancel{t}^2}{2}$

$f''(\cancel{y}) = \frac{d}{dt}\frac{y}{8}(6-y) = \frac{-y}{8} + \frac{6-y}{8} = \frac{3-y}{4}$

$8 - 2x - \frac{3-y}{4} = y$

$6 - \frac{3-y}{4} = y$

$24 - 3 + y = 4y$

$21 = 3y$

$y = 7 = f(1)$

Work for problem 6(d)

Asymptotes at: $\frac{y}{8}(6-y)=0$, $y=0$, $y=6$. All other values are achievable.

Thus the range is $\{y \mid y \in \mathbb{R}, y \neq 0, 6\}$

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Question 6

Overview

This problem presented students with a logistic differential equation and the initial value $f(0) = 8$ of a particular solution $y = f(t)$. In part (a) a slope field for the differential equation was given, and students were asked to sketch solution curves through two specified points. In particular, students should have demonstrated appropriate behavior for these curves for $t \geq 0$, especially with regard to the horizontal lines $y = 0$ and $y = 6$. For part (b) students needed to use the given initial value for the solution f and a two-step Euler's method to approximate $f(1)$. In part (c) students were directed to find the second-degree Taylor polynomial for f about $t = 0$ and use it to approximate $f(1)$. Part (d) asked for the range of the particular solution $y = f(t)$.

Sample: 6A
Score: 9

The student earned all 9 points. Note that the reason provided by the student in part (d) was not required.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 3 points in part (c), and no points in part (d). In part (a) the student presents a solution curve through the point $(0, 8)$ and a solution curve through the point $(3, 2)$. Both curves presented have the general form required relative to the slope field, and the student earned both points. In part (b) the student earned the first point for correctly demonstrating two steps of Euler's method using the initial value $f(0) = 8$ and a step size of $\Delta t = \frac{1}{2}$ to approximate $f(1)$. The student makes a copy error in the second step of Euler's method when $y = 9$ is used instead of $y = 7$ in the first term of $\frac{dy}{dt}$; therefore, the response did not earn the second point. In part (c) the first 2 points were earned when the student correctly finds $\frac{d^2y}{dt^2}$. The student goes on to evaluate $f'(0)$ and $f''(0)$. The student earned the third point when using the declared values of $f'(0)$ and $f''(0)$ along with $f(0) = 8$ to correctly construct a second-degree Taylor polynomial for f about $t = 0$. The student makes an arithmetic error in the calculation of $f''(0)$ so did not earn the fourth point. In part (d) the student presents an incorrect range for f so did not earn the point.

Sample: 6C
Score: 4

The student earned 4 points: 2 points in part (a), 2 points in part (b), no points in part (c), and no points in part (d). In part (a) the student presents a solution curve through the point $(0, 8)$ and a solution curve through the point $(3, 2)$. Both curves presented have the general form required relative to the slope field, and the student earned both points. In part (b) the student correctly demonstrates two steps of Euler's method using the initial value $f(0) = 8$ and a step size of $\Delta t = \frac{1}{2}$ to approximate $f(1)$. The student earned both points. Note that the student was not required to provide a decimal presentation of the approximation to $f(1)$ but does so correctly. In part (c) the student

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Question 6 (continued)

does not use the chain rule to find $\frac{d^2y}{dt^2}$ and did not earn the points. The student provides a formula for the second-degree Taylor polynomial but does not calculate values of $f'(0)$ and $f''(0)$ to construct a Taylor polynomial that could be used to approximate $f(1)$. The student was not eligible for the third point and, as a result, was not eligible for the fourth point. In part (d) the student presents an incorrect range for f so did not earn the point.

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